higher pressure when the time between collisions is decreased. As the density is decreased, the time that a particle is acted upon by another, compared to the time between collisions, will decrease in a manner proportional to the density. Therefore, the interparticle forces may probably be neglected for the lower densities. Magee and Heller[11] used a Debye-Huckel ionic solution in estimating the effect of the ionic forces on helium, among other plasmas, up to 50,000°K and found that this correction amounted to only a few percent at most. It is felt that the same results apply to a higher temperature, probably with a somewhat higher error. However, at this time there exists, to the knowledge of the authors, no adequate expression for these forces which would allow one to compensate for their effects with an accuracy justifying the additional complications.

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A Semitoroidal Reflex Discharge as a Propulsion Device

D. PIGACHE*

Stanford University, Stanford, Calif.

A REFLEX discharge consists essentially of two cathodes situated at each end of a cylindrical hollow anode. An axial magnetic field confines the plasma that is formed by ionizing electrons emitted by the cathodes and accelerated by the anode. Because of the symmetrical variation of the potential, the electrons are first accelerated by the anode, cross the anode region (almost equipotential), and then are reflected by the second cathode. The electrons may oscillate back and forth between cathodes several times and can lose an important part of their energy in ionizing collisions with neutrals before falling finally to the anode. The ions formed in the preceding collisions are accelerated by the negative potential of one or the other of the cathodes. If Vd is the voltage of the discharge, the energy acquired by the ion as it falls to the cathode is on the order of eV_d. If one of the cathodes has a ring shape,[1] the ions can travel through it. The beam of ions extracted is automatically neutralized by electrons that have sufficient energy to cross the potential barrier of the cathode and thus accompany the ions outside the discharge region.

The electrons that do not accompany the ions are reflected by the voltage drop that occurs in the ring cathode region, and they behave exactly as in an ordinary reflex discharge. Therefore, the high degree of ionization provided by the combined electric and magnetic field of the reflex discharge is not affected.

Electrical efficiency η_E is defined as the ratio of the kinetic energy of the extracted plasma beam to the total electrical energy expended in the system

$$\eta_E = P_C/P_E$$

where

P_C = kinetic power of beam
P_E = electrical power expended

If A is the number of ionizing collisions produced by one electron emitted by the cathode and accelerated by the cathode-anode potential, and K is the number of ions lost (falling on cathodes) divided by the total number of ions produced, then it can be shown[2] that

$$\eta_E = A(1 - K)/(1 + A)$$

One can see immediately that if $A \to \infty$ (i.e., if the ionization energy becomes negligible), $\eta_E \to 1 - K$. For a linear discharge terminated at one end by the emitting cathode, $K \geq 0.5$, giving $\eta_E \leq 50\%$. This results from the fact that the ions formed in the anode region have equal chance to be extracted or to fall on the emitting cathode where their kinetic energy is lost. Then it seemed at first that the efficiency of a reflex discharge as a propulsion device was basically limited to 50%.

However, the addition of a mirror magnetic field forces the electrostatic potential to be asymmetric.[3] As a result, the ions are directed toward the exit rather than toward the emitting cathode.

A reflex discharge having a semitoroidal shape, and using two cylindrical emitting cathodes, offers another possibility to increase the efficiency above 50%. The diagram of the discharge is given in Fig. 1.

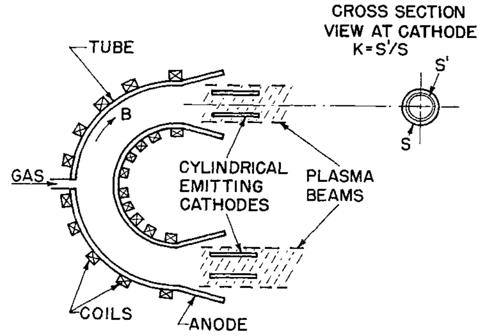

Fig. 1 Diagram of semitoroidal discharge (the tube may be of insulating material or may be metal and be at anode potential).

In that case, K has the lower theoretical value of $K = S'/S$ where S is the area of the torus cross section (perpendicular to the torus axis), and S' is the area of the cathode cross section (perpendicular to the axis). A typical value of K is $K = 0.1$. The ionization efficiency of the electrons is conveniently

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* Research Associate, Stanford Electronics Laboratories.

Table 1 Values of A as a function of the electron energy in argon and hydrogen

	Electron energy, ev				
	30	50	100	150	200
Argon	0.45	0.9	1.6	2.2	...
Hydrogen	...	...	1.4	...	2.8

estimated by use of the parameter A, whose maximum value corresponds to the number of ionizing collisions made by an electron of a given energy when all the energy is expended in the gas. This number has been measured experimentally for different gases.[4]
For energies above 200 ev, A is almost proportional to the energy of the electron.

One can show[5,6] that, in a reflex discharge, A is slightly lower than the maximum value given in Table 1. Typical values of the discharge voltage are 150 or 200 v. With $A = 2$ and $K = 0.1$, it is found that $\eta_E = 0.6$.

No experiments have been performed on a semitoroidal discharge, but the operation of a complete toroidal reflex discharge using oxide cathodes identical to those shown in Fig. 1 has been observed experimentally.[8,9] Plasma densities attained in the toroidal discharge and in the linear discharge, using the same type of cathodes,[5-7] are of the order of 5×10^{13} electrons/cm^3 for a discharge current of 100 amp. Ionization rates are higher than 50%.

Efficiency will increase with higher discharge voltage. Discharge currents of 60 amps with voltage of 500 v have been obtained recently on a toroidal discharge with good stability. The cathode was operating with reduced emission. For this voltage, one would expect the value of A to be of the order of $A \simeq 8$.[4] For $A = 8, K = 0.1, \eta_E = 0.80$.

Discussion

A semitoroidal reflex discharge might have an efficiency comparable to that of an ionic propulsion device. Although the good operation of a toroidal reflex discharge has already been established, some questions remain unanswered.

1) The ions can be attracted by the negative potential of the cathode and made to fall on it. Therefore, K could be somewhat higher than S'/S.

2) It has been assumed that the losses are only due to ions falling to the cathodes. It is not sure that the losses in the torus, by perpendicular diffusion across the magnetic field and recombination on the wall, can be reduced to an insignificant amount. However, an encouraging result obtained on the toroidal discharge is the fact that the rate of loss measured experimentally[10] is in good agreement with the calculated value of loss rate taking into account classical diffusion due to collisions and drift losses to the curvature of the magnetic field. The drift losses can be at least partially eliminated by use of a bumpy magnetic field.[8] Anomalous diffusion does not seem to occur in our discharge for magnetic field up to 600 gauss (maximum available). However, anomalous diffusion occurs in a linear discharge for magnetic fields higher than 1000 gauss[7] and might occur also in the toroidal discharge. But high magnetic fields are not desirable for propulsion device for other reasons, i.e., loss of power in the coils and increase of weight.

3) What will be the ratio of useful propellant to propellant consumed? It is difficult to predict a priori, but it seems that the efficiency of ionization of the electrons in the reflex discharges that we have studied is at least as good as in those described by Kaufman and Reader[11] in which ratios up to 0.8 (80%) were obtained.

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Derivation of Element Stiffness Matrices by Assumed Stress Distributions

THEODORE H. H. PIAN*
Massachusetts Institute of Technology, Cambridge, Mass.

IN a recent note,[1] the author has outlined a procedure for evaluating the element stiffness matrices to be used in connection with the displacement method of matrix structural analyses. The procedure is based on the representation of the displacement of a structural element in terms of displacement functions of m undetermined coefficients. In general, in order to satisfy the equilibrium of stress in the interior, m should be larger than the number of the generalized displacements n. When m and n are equal, these undetermined coefficients can be directly related to the generalized displacements. When m is larger than n, they can be evaluated by the employment of the principle of minimum potential energy. An important requirement is that the displacement functions must maintain compatibility with the adjacent elements. This condition has been emphasized by many previous authors.[2,3]

This approach of assumed displacement functions is particularly suitable for one-dimensional elements, such as segments of axisymmetrical shells,[4] for which the displacement compatibility with the adjacent elements is completely satisfied when the corresponding generalized displacements coincide at the nodes. For two-dimensional problems such as general shells or plates under bending or in plane stress conditions, it is not always a straightforward matter to write down a displacement function that will yield compatible boundary displacements. For example, Melosh[2] has presented an expression for the bending displacement of a rectangular plate. That expression only provides continuity of displacements at all edges, but it will not maintain continuity of slopes along the normal directions of the four edges. The present author feels that a completely compatible boundary displacement for bending of plates and shells should include the slope continuity as well.

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* Associate Professor of Aeronautics and Astronautics. Associate Fellow Member AIAA.